Tutorial 4: Velocity Kinematics

These questions are from the Practice Exercises of the Modern Robotics book. The solutions can be found on the book website. Please try your best before referring to the solutions.

Source of questions, solutions and images: Practice Exercises of the Modern Robotics.

Question 1: KUKA LBR iiwa 7R robot arm

Figure 1 shows the KUKA LBR iiwa 7R robot arm. The figure defines an $\{s\}$ frame at the base with the $\hat{y}_s$-axis pointing out of the page and a $\{b\}$ frame aligned with $\{s\}$ at the end-effector. The robot is at its home configuration. The screw axes for the seven joints are illustrated (positive rotation about these axes is by the right-hand rule). The axes for joints 2, 4, and 6 are aligned, and the axes for joints 1, 3, 5, and 7 are identical at the home configuration. The dimensions are $L_1 = 0.34\ m$, $L_2 = 0.4\ m$, $L_3 = 0.4\ m$, and $L_4 = 0.15\ m$.

a. What is the space Jacobian when the robot is at its home configuration?
b. Assume the angles of the joints are ${}^{i\pi}/_{16}$ radians for joints $i = 1 \cdots 7$. What is the space Jacobian?

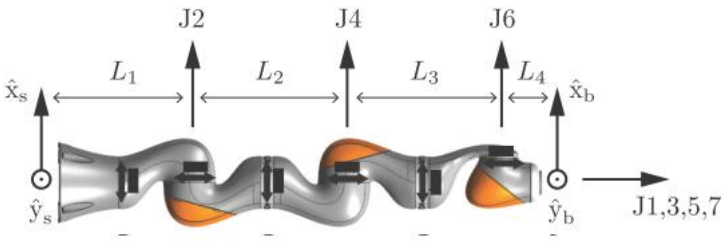


Figure 1. The KUKA LBR iiwa 7-dof robot. Note $\widehat{y_s}$ and $\widehat{y_b}$ are pointing out of the paper.

Question 2: RPR robot arm

Figure 2 shows an RPR robot that is confined to the plane of the page. An end-effector frame $\{b\}$ is illustrated, where the $\hat{x}_b$-axis is out of the page. The directions of positive motion of the three joints are indicated by arrows. The axes of the two revolute joints are out of the page, and the prismatic joint moves in the plane of the page. Joint 1 is at $q_1 = (0, -5,7)$ in $\{b\}$ and joint 3 is at $q_3 = (0, -1, -3)$ in $\{b\}$. Write the body Jacobian $J_b(\theta)$ for the configuration shown. All entries of your $J_b(\theta)$ matrix should be numerical (no symbols or math).

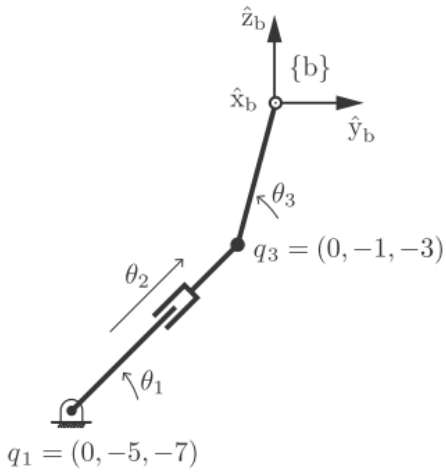


Figure 2. An RPR Robot.

Sample solution

Ref: Modern Robotics Practice Exercises.

Question 1: KUKA LBR iiwa 7R robot arm

The sample solutions are purposely elaborated to show the calculations involved. They have been manually worked out and have not been verified. The sample answers may have errors; they are meant to show the steps involved. The sample solution also presented multiple ways to solve the problems; you just need to use any one of the approaches.

In real scenarios, these calculations are done on a computer. However, for the purpose of the assessment in this module, you are expected to be able to perform the calculations manually for simple robotic arms.

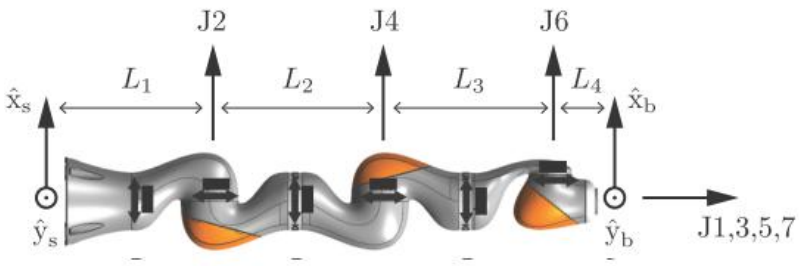
First note the robot arm has 7 joints, so n=7, the Jacobian matrix will have 7 columns.

$$J_s(\boldsymbol{\theta}) = \begin{bmatrix} J_{s1}(\boldsymbol{\theta}) & J_{s2}(\boldsymbol{\theta}) & J_{s3}(\boldsymbol{\theta}) & J_{s4}(\boldsymbol{\theta}) & J_{s5}(\boldsymbol{\theta}) & J_{s6}(\boldsymbol{\theta}) & J_{s7}(\boldsymbol{\theta}) \end{bmatrix}$$

Each column of the Jacobian matrix can be determined as a screw axis and linear velocity around the screw axis similar to the way we did in forward kinematics. However, the screw axis and linear velocity must take in consideration of arbitrary joint angles, i.e. not just at home configuration.

$$J_{s1}(\boldsymbol{\theta}) = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \omega_{x1} \\ \omega_{y1} \\ \omega_{z1} \\ v_{x1} \\ v_{y1} \\ v_{z1} \end{bmatrix}, J_{s2}(\boldsymbol{\theta}) = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \omega_{x2} \\ \omega_{y2} \\ \omega_{z2} \\ v_{x2} \\ v_{y2} \\ v_{z2} \end{bmatrix}, J_{s3}(\boldsymbol{\theta}) = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \omega_{x3} \\ \omega_{y3} \\ \omega_{z3} \\ v_{x3} \\ v_{y3} \\ v_{z3} \end{bmatrix}, \text{etc}$$

We take note of the whereabout of J1 to J7.



Since we are interested in space Jacobian, we begin with the joint closest to the {s} frame, i.e. J1 and progress further away from {s}, i.e. J2, J3, .. J7. Taking note that as we move to further joints, the motion of preceding joints will affect the current joint being considered. E.g. for J4, the ω_4 and v_4 will be affected by the values of θ_1, θ_2 and θ_3, i.e. motion of preceding joints.

- $J_{s1}(\theta)$ is not affected by any preceding joints, so will be determined as at home configuration.
- $J_{s2}(\theta)$ is subject to rotation of preceding joint θ_1, so it will be in terms of θ_1.
- $J_{s3}(\theta)$ is subject to rotation of preceding joints θ_1 and θ_2, so it will be in terms of θ_1 and θ_2.
- $J_{s4}(\theta)$ is subject to rotation of preceding joints θ_1, θ_2 and θ_3, so it will be in terms of θ_1, θ_2 and θ_3.
- And so forth.

Two approaches:

1. We may derive a general expression for each column of the Jacobian matrix in terms of $\boldsymbol{\theta}$. The expression can be used to determine the Jacobian matrix given any values of $\boldsymbol{\theta}$. For (a), we substitute $\boldsymbol{\theta} = [0]$. For (b), we substitute $\boldsymbol{\theta} = \left[\theta_i = {}^{i\pi}/_{16}\ for\ i = 1 \dots 7\right]$.
2. Alternatively, we can use the values of $\boldsymbol{\theta}$ to determine each ω_i and v_i from the beginning without trying to find the general expression. We can use rotation matrices to account for the rotation induced by earlier joints.

<u>Question 1(a): Exercise 5.1(a)</u>

The sample solution below uses the **second approach**.

Note this question determines the Jacobian at a specific configuration, the zero/home configuration at $\boldsymbol{\theta} = [0]$. Therefore, the final answer of the Jacobian is not a function of $\boldsymbol{\theta}$.

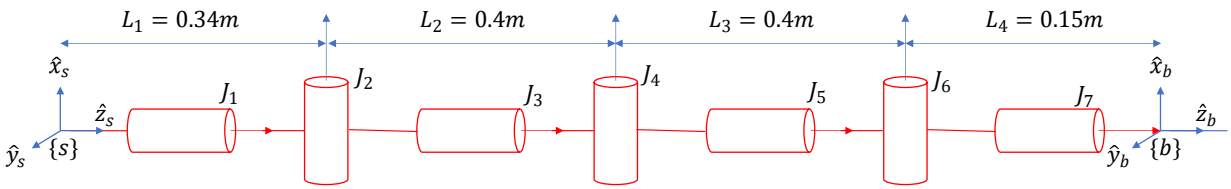


Figure 3. Zero/home configuration.

At home configuration,

$$J_s^0 = [J_{s1}^0 \quad J_{s2}^0 \quad J_{s3}^0 \quad J_{s4}^0 \quad J_{s5}^0 \quad J_{s6}^0 \quad J_{s7}^0]$$

We will determine each Jacobian column.

$$J_{s1}^0 = \begin{bmatrix} \omega_1^0 \\ v_1^0 \end{bmatrix} \qquad \omega_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ in the direction of } J_1\text{, i.e. } \hat{z}_s.$$

$$v_1^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ no linear motion on \{s\}.}$$

$$J_{s2}^0 = \begin{bmatrix} \omega_2^0 \\ v_2^0 \end{bmatrix} \qquad \omega_2^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ in the direction of } J_2\text{, i.e. } \hat{x}_s.$$

$$v_2^0 = \begin{bmatrix} 0 \\ L_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.34 \\ 0 \end{bmatrix}$$

when rotate around J_2, {s} experiences a linear motion in the direction of $\hat{y}_s$ at a distance of L_1.

$$J_{s3}^0 = \begin{bmatrix} \omega_3^0 \\ v_3^0 \end{bmatrix} \qquad \omega_3^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ in the direction of } J_3\text{, i.e. } \hat{z}_s.$$

$$v_3^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rotating around J_3 does not cause linear motion on {s}.

$$J_{s4}^0 = \begin{bmatrix} \omega_4^0 \\ v_4^0 \end{bmatrix} \qquad \omega_4^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ in the direction of } J_4\text{, i.e. } \hat{x}_s.$$

$$v_4^0 = \begin{bmatrix} 0 \\ L_1 + L_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.74 \\ 0 \end{bmatrix}$$

rotating around J_4 causes {s} to move (linear) in the direction of $\hat{y}_s$ according to RHR (right hand rule), at a distance of $L_1 + L_2$.

$$J_{s5}^0 = \begin{bmatrix} \omega_5^0 \\ v_5^0 \end{bmatrix} \qquad \omega_5^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_5^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

similar to J_3.

$$J_{s6}^0 = \begin{bmatrix} \omega_6^0 \\ v_6^0 \end{bmatrix} \qquad \omega_6^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_6^0 = \begin{bmatrix} 0 \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.14 \\ 0 \end{bmatrix}$$

similar to J_4.

$$J_{s7}^0 = \begin{bmatrix} \omega_7^0 \\ v_7^0 \end{bmatrix} \qquad \omega_7^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_7^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

similar to J_1, J_3, J_5.

Question 1(b): Exercise 5.1(b)

Let us take the **first approach** to obtain a general expression for $J_s(\boldsymbol{\theta})$ in terms of the joint variables $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7)$. Note we could solve 1(a) by first deriving the genearla expression for $J_s(\boldsymbol{\theta})$ and then substitute $\boldsymbol{\theta} = (0,0,0,0,0,0,0)$. However, you will notice deriving a general expression by hand can get complicated with many joints.

One way to find ω_i and v_i at an arbitrary angle $\boldsymbol{\theta}$ is to apply the transformation caused by $\theta_{i-1} \dots \theta_1$, i.e. preceding joints.

We want to determine

$$J_s(\boldsymbol{\theta}) = \begin{bmatrix} J_{s1}(\boldsymbol{\theta}) & J_{s2}(\boldsymbol{\theta}) & J_{s3}(\boldsymbol{\theta}) & J_{s4}(\boldsymbol{\theta}) & J_{s5}(\boldsymbol{\theta}) & J_{s6}(\boldsymbol{\theta}) & J_{s7}(\boldsymbol{\theta}) \end{bmatrix}$$

We will determine each Jacobian column.

Determine $J_{s1}(\boldsymbol{\theta})$

J1 is the direction of $\hat{z}_s$, and is not affected by the rotation of any other joints and rotation around ω_1 does not induce any linear motion.

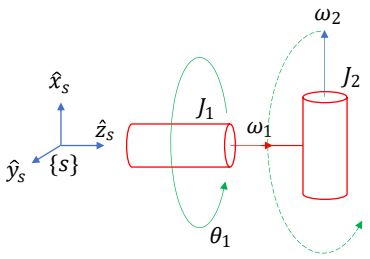
$$J_{s1} = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix}$$

The axis of J_1, i.e. ω_1 is not affected by any θ_i, it remains at $\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Likewise, rotating around it does not cause any linear motion on {s}, $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$J_{s1}(\theta) = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

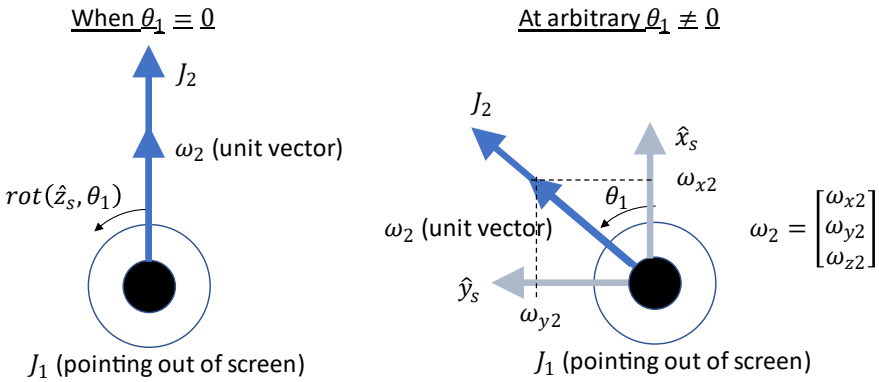
Determine $J_{s2}(\boldsymbol{\theta})$



Notice the direction of J_2, i.e. ω_2 is affected by (dependent on) θ_1 (preceding joint).

J2 is in the direction of $\hat{x}_s$ at home configuration. However this direction will change when θ_1 changes. $J_{s2}(\theta)$ will be a function of θ_1. We can determine the direction of J2, i.e. ω_2, from **(1)** the geometry **or** by **(2)** applying the rotation of θ_1 on the home configuration of ω_2.

(1) By geometry



$$\omega_2 = \begin{bmatrix} \omega_{x2} \\ \omega_{y2} \\ \omega_{z2} \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{bmatrix}$$

(2) By applying rotation of θ_1 on ω_2 at home configuration.

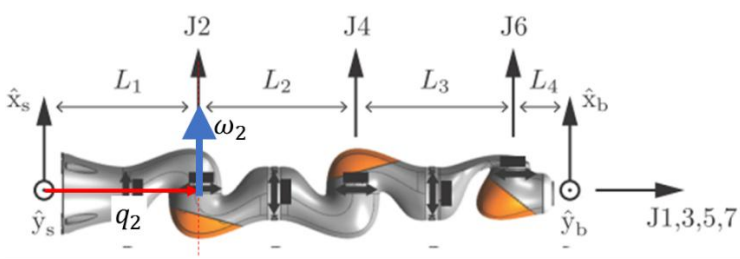
Note at home configuration, ${\omega_2}^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in the direction of $\hat{x}_s$, and θ_1 rotate around $\hat{z}_s$.

We can determine ω_2 by applying the rotation of θ_1 on ${\omega_2}^0$.

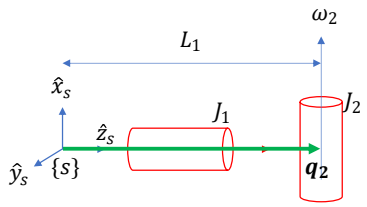
$$\begin{aligned} \omega_2 &= rot(\hat{z}_s, \theta_1){\omega_2}^0 \\ &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{bmatrix} \end{aligned}$$

Rotating around ω_2 will induce a linear motion at the origin of {s}. If ω_2 was at home configuration, the linear motion will be in the direction of the $\hat{y}_s$. However, for Jacobian, we need to account for the changes to ω_2 due to θ_1.

We determine the linear velocity v_2 by noting that it is perpendicular to ω_2 and a vector q_2 connect the origin of {s} to ω_2.



(at this configuration, v_2 is along $\hat{y}_s$)



$$\boldsymbol{q_2} = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$$

(expressed in {s})

Given the direction of q_2 we have defined, we use cross product to find v_2

$$v_2 = -\omega_2 \times q_2$$

Note q_2 is in the direction of $\hat{z}_s$ (even if θ_1 changes) with a length of L_1. We have

$$\omega_2 = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{bmatrix} \; and \; q_2 = \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix}$$

Giving

$$v_2 = -\omega_2 \times q_2 = -\begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 \\ L_1 \cos\theta_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -L_1 s_1 \\ L_1 c_1 \\ 0 \end{bmatrix} = L_1 \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

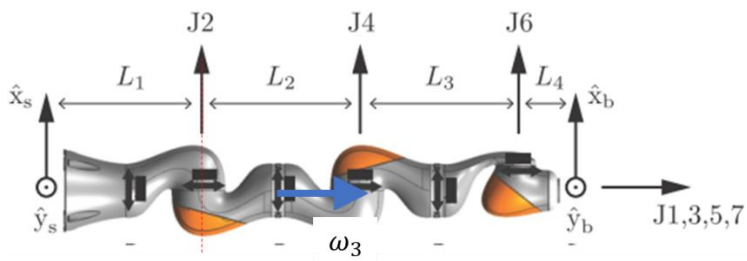
Note $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$.

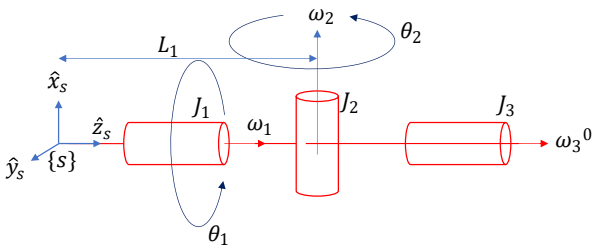
Now we have

$$J_{s2}(\theta) = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \\ -L_1 \sin\theta_1 \\ L_1 \cos\theta_1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ s_1 \\ 0 \\ -L_1 s_1 \\ L_1 c_1 \\ 0 \end{bmatrix}$$

<u>Determine $J_{s3}(\boldsymbol{\theta})$</u>

We move on to determine $J_{s3}(\theta)$.





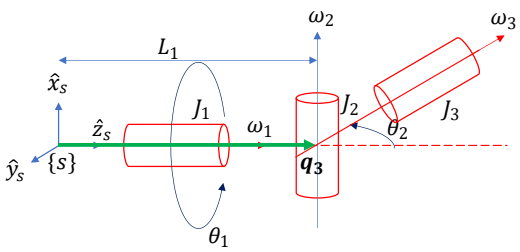
We considering ω_3 and v_3, we note they are transformed from $\omega_3{}^0$ and $v_3{}^0$ by θ_1 and θ_2.

We note $\omega_3{}^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. However, the direction of ω_3 will change by preceding joints i.e. θ_1 and θ_2. It gets difficult to visualize by geometry. We will determine ω_3 by applying rotation of θ_1 and θ_2. To ensure we will rotate around the axes of the space frame, we apply the rotation on J_n in the order of rotation at J_{n-1}, then rotation at J_{n-2}, and so forth. In this way, we are rotating with reference to a fix frame {s}, and we will perform pre-multiplication.

For J_3, we apply the rotation around J_2, followed by rotation around J_1. In other words, we apply on $\omega_3{}^0$ a rotation about the $\hat{x}_s$ (J_2) by θ_2 followed by a rotation about the $\hat{z}_s$ (J_1) by θ_1. Since these are rotations with reference to a fix frame {s}, we perform pre-multiplication.

$$
\begin{aligned}
\omega_3 &= rot(\hat{z}_s, \theta_1) rot(\hat{x}_s, \theta_2) \omega_3{}^0 \\
&= R_1 rot(\hat{x}_s, \theta_2) \omega_3{}^0 \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1\cos\theta_2 & \sin\theta_1\sin\theta_2 \\ \sin\theta_1 & \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2 \\ 0 & \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} s_1 s_2 \\ -c_1 s_2 \\ c_2 \end{bmatrix}
\end{aligned}
$$

We determine v_3 by cross product.



Note q_3 is defined from {s} to the intersection of ω_2 and ω_3. Basically q_3 can be from {s} to any point on ω_3. Since ω_3 intersects at ω_2, this point is convenient. In this case, q_3 is the same as q_2.

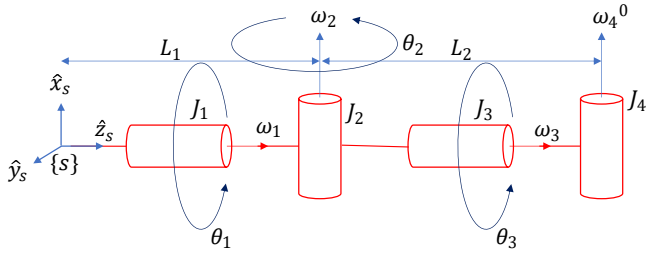
$$
\begin{aligned}
v_3 &= -\omega_3 \times q_3 \\
&= -\begin{bmatrix} s_1 s_2 \\ -c_1 s_2 \\ c_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} \\
&= -\begin{bmatrix} -L_1 c_1 s_2 \\ -L_1 s_1 s_2 \\ 0 \end{bmatrix} \\
&= L_1 \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ 0 \end{bmatrix}
\end{aligned}
$$

Now we have

$$J_{s3}(\theta) = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} s_1 s_2 \\ -c_1 s_2 \\ c_2 \\ L_1 c_1 s_2 \\ L_1 s_1 s_2 \\ 0 \end{bmatrix}$$

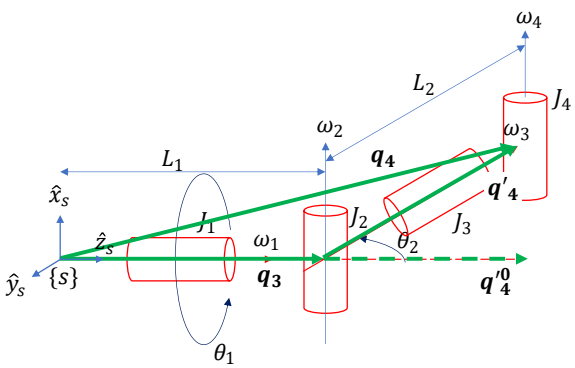
<u>Determine $J_{s4}(\boldsymbol{\theta})$</u>

For J_{s4},



$$\begin{aligned}
\omega_4 &= rot(\hat{z}_s, \theta_1) rot(\hat{x}_s, \theta_2) rot(\hat{z}_s, \theta_3) \omega_4{}^0 \\
&= R_2 rot(\hat{z}_s, \theta_3) \omega_4{}^0 \\
&= \begin{bmatrix} c_1 & -s_1 c_2 & s_1 s_2 \\ s_1 & c_1 c_2 & -c_1 s_2 \\ 0 & s_2 & c_2 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 & -c_1 s_3 - s_1 c_2 c_3 & s_1 s_2 \\ s_1 c_3 + c_1 c_2 s_3 & -s_1 s_3 + c_1 c_2 c_3 & -c_1 s_2 \\ s_2 s_3 & s_2 c_3 & c_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = R_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 \\ s_1 c_3 + c_1 c_2 s_3 \\ s_2 s_3 \end{bmatrix}
\end{aligned}$$

As with move to J_4, v_4 is subject to the rotations of θ_1, θ_2 and θ_3. q_4 will be a function of θ_1, θ_2 and θ_3. Determining it is getting difficult. We note we can set q_4 at the intersection of ω_3 and ω_4. With this setting, q_4 will be a function of θ_1 and θ_2.



$$q_4 = q_3 + q'_4$$

We take note of the rotation effect of θ_1 and θ_2 on $q'_4{}^0$.

$$q'_4 = rot(\hat{z}_s, \theta_1) rot(\hat{x}_s, \theta_2) q'_4{}^0$$

$$\begin{aligned}
q_4 &= q_3 + q'_4 \\
&= q_3 + rot(\hat{z}_s, \theta_1) rot(\hat{x}_s, \theta_2) q'_4{}^0 \\
&= q_3 + R_2 q'_4{}^0
\end{aligned}$$

$$= \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + \begin{bmatrix} c_1 & -s_1c_2 & s_1s_2 \\ s_1 & c_1c_2 & -c_1s_2 \\ 0 & s_2 & c_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ L_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ L_1 \end{bmatrix} + \begin{bmatrix} L_2s_1s_2 \\ -L_2c_1s_2 \\ L_2c_2 \end{bmatrix}$$

$$= \begin{bmatrix} L_2s_1s_2 \\ -L_2c_1s_2 \\ L_1 + L_2c_2 \end{bmatrix}$$

$$v_4 = -\omega_4 \times q_4$$

$$= -\begin{bmatrix} c_1c_3 - s_1c_2s_3 \\ s_1c_3 + c_1c_2s_3 \\ s_2s_3 \end{bmatrix} \times \begin{bmatrix} L_2s_1s_2 \\ -L_2c_1s_2 \\ L_1 + L_2c_2 \end{bmatrix}$$

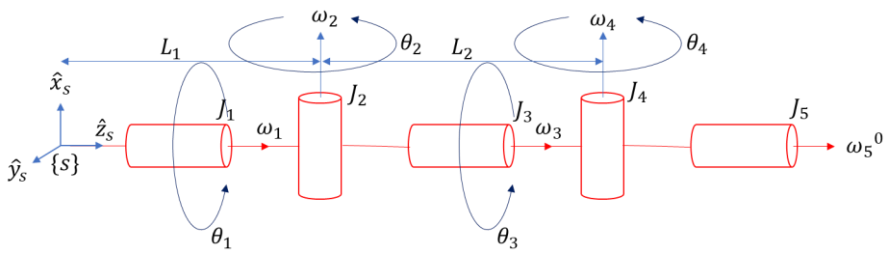
$$= -\begin{bmatrix} L_1(s_1c_3 + c_1c_2s_3) + L_2(s_1c_2c_3 + c_1s_3) \\ L_1(-c_1c_3 + s_1c_2s_3) + L_2(s_1s_3 - c_1c_2c_3) \\ -L_2s_2c_3 \end{bmatrix}$$

Now we have

$$J_{s4}(\theta) = \begin{bmatrix} \omega_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} c_1c_3 - s_1c_2s_3 \\ s_1c_3 + c_1c_2s_3 \\ s_2s_3 \\ -L_1(s_1c_3 + c_1c_2s_3) - L_2(s_1c_2c_3 + c_1s_3) \\ L_1(c_1c_3 - s_1c_2s_3) + L_2(c_1c_2c_3 - s_1s_3) \\ L_2s_2c_3 \end{bmatrix}$$

<u>Determine $J_{s5}(\boldsymbol{\theta})$</u>

For J_{s5},



$$\omega_5 = rot(\hat{z}_s, \theta_1)rot(\hat{x}_s, \theta_2)rot(\hat{z}_s, \theta_3)rot(\hat{x}_s, \theta_4)\omega_5{}^0$$

$$= R_3 rot(\hat{x}_s, \theta_4)\omega_5{}^0$$

$$= \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3 - s_1c_2c_3 & s_1s_2 \\ s_1c_3 + c_1c_2s_3 & -s_1s_3 + c_1c_2c_3 & -c_1s_2 \\ s_2s_3 & s_2c_3 & c_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_4 & -s_4 \\ 0 & s_4 & c_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3c_4 - s_1c_2c_3c_4 + s_1s_2s_4 & c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1c_3 + c_1c_2s_3 & -s_1s_3c_4 + c_1c_2c_3c_4 - c_1s_2s_4 & s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ s_2s_3 & s_2c_3c_4 + c_2s_4 & -s_2c_3s_4 + c_2c_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = R_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

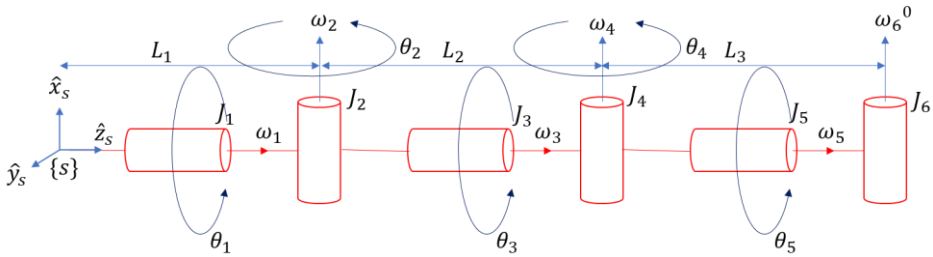
$$= \begin{bmatrix} c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ -s_2c_3s_4 + c_2c_4 \end{bmatrix}$$

Since ω_5 intersects with ω_4, we can use $q_5 = q_4$. This is getting messy to expand. We will leave v_5 in this state. It is easier to substitute the values of θ_i and compute the cross product.

$$
\begin{aligned}
v_5 &= -\omega_5 \times q_5 \\
&= -\begin{bmatrix} c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix} \times \begin{bmatrix} L_2 s_1 s_2 \\ -L_2 c_1 s_2 \\ L_1 + L_2 c_2 \end{bmatrix}
\end{aligned}
$$

<u>Determine $J_{s6}(\boldsymbol{\theta})$</u>

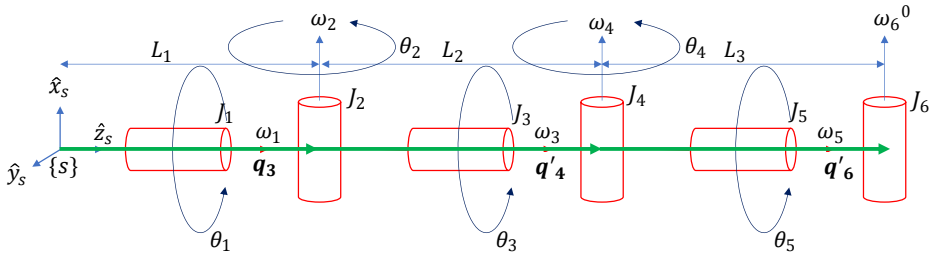
For J_{s6},



$$
\begin{aligned}
\omega_6 &= R_4 rot(\hat{z}_s, \theta_5)\omega_6{}^0 \\
&= R_4 \begin{bmatrix} c_5 & -s_5 & 0 \\ s_5 & c_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= R_4 \begin{bmatrix} c_5 \\ s_5 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 & -c_1 s_3 c_4 - s_1 c_2 c_3 c_4 + s_1 s_2 s_4 & c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 c_3 + c_1 c_2 s_3 & -s_1 s_3 c_4 + c_1 c_2 c_3 c_4 - c_1 s_2 s_4 & s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ s_2 s_3 & s_2 c_3 c_4 + c_2 s_4 & -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix} \begin{bmatrix} c_5 \\ s_5 \\ 0 \end{bmatrix}
\end{aligned}
$$

We will leave ω_6 in the above state. Expanding the matrix multiplication will be messy. We will substitute the angle values where necessary to compute ω_6 for a given set of joint angles.

Using the same geometry concepts we did for q_4, we can determine q_6.



$$q_6 = q_3 + q'_4 \quad + +q'_6 \quad = q_4 + q'_6 \quad = q_4 + rot(\hat{z}_s, \theta_3) rot(\hat{x}_s, \theta_4) q'_6{}^0 = q_4 + R_2(\theta_3, \theta_4) q'_6{}^0$$

Note we are reusing R_2 from the previous steps, replacing the angles accordingly.

$$
\begin{aligned}
q_6 &= q_4 + R_2(\theta_3, \theta_4) q'_6{}^0 \\
&= \begin{bmatrix} L_2 s_1 s_2 \\ -L_2 c_1 s_2 \\ L_1 + L_2 c_2 \end{bmatrix} + \begin{bmatrix} c_3 & -s_3 c_4 & s_3 s_4 \\ s_3 & c_3 c_4 & -c_3 s_4 \\ 0 & s_4 & c_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ L_3 \end{bmatrix}
\end{aligned}
$$

$$= \begin{bmatrix} L_2 s_1 s_2 \\ -L_2 c_1 s_2 \\ L_1 + L_2 c_2 \end{bmatrix} + \begin{bmatrix} L_3 s_3 s_4 \\ -L_3 c_3 s_4 \\ L_3 c_4 \end{bmatrix}$$
$$= \begin{bmatrix} L_2 s_1 s_2 + L_3 s_3 s_4 \\ -L_2 c_1 s_2 - L_3 c_3 s_4 \\ L_1 + L_2 c_2 + L_3 c_4 \end{bmatrix}$$

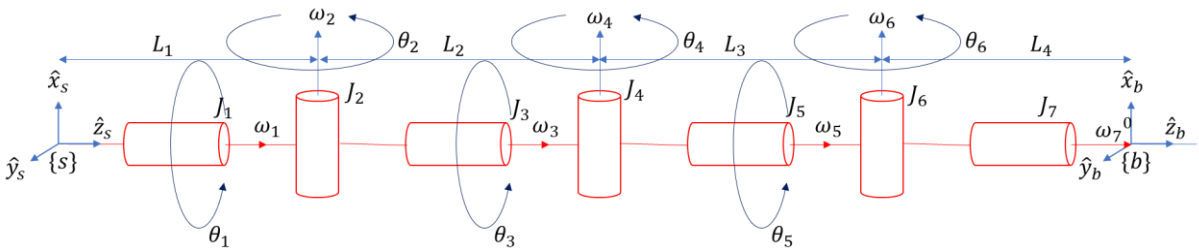
We have

$$v_6 = -\omega_6 \times q_6$$
$$= -\omega_6 \times \begin{bmatrix} L_2 s_1 s_2 + L_3 s_3 s_4 \\ -L_2 c_1 s_2 - L_3 c_3 s_4 \\ L_1 + L_2 c_2 + L_3 c_4 \end{bmatrix}$$

Let's leave v_6 in the above state.

<u>Determine $J_{s7}(\boldsymbol{\theta})$</u>

For J_{s7},



$$\omega_7 = R_4 rot(\hat{z}_s, \theta_5) rot(\hat{x}_s, \theta_6) \omega_7{}^0$$
$$= R_4 \begin{bmatrix} c_5 & -s_5 & 0 \\ s_5 & c_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_6 & -s_6 \\ 0 & s_6 & c_6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= R_4 \begin{bmatrix} c_5 & -s_5 c_6 & s_5 s_6 \\ s_5 & c_5 c_6 & -c_5 s_6 \\ 0 & s_6 & c_6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= R_4 \begin{bmatrix} s_5 s_6 \\ -c_5 s_6 \\ c_6 \end{bmatrix}$$
$$= \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 & -c_1 s_3 c_4 - s_1 c_2 c_3 c_4 + s_1 s_2 s_4 & c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 c_3 + c_1 c_2 s_3 & -s_1 s_3 c_4 + c_1 c_2 c_3 c_4 - c_1 s_2 s_4 & s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ s_2 s_3 & s_2 c_3 c_4 + c_2 s_4 & -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix} \begin{bmatrix} s_5 s_6 \\ -c_5 s_6 \\ c_6 \end{bmatrix}$$

We will leave ω_7 in the above state. Since ω_7 intersects with ω_6, we can use $q_7 = q_6$.

$$v_7 = -\omega_7 \times q_7$$
$$= -\omega_7 \times \begin{bmatrix} L_2 s_1 s_2 + L_3 s_3 s_4 \\ -L_2 c_1 s_2 - L_3 c_3 s_4 \\ L_1 + L_2 c_2 + L_3 c_4 \end{bmatrix}$$

We will leave v_7 in the above state.

We have obtained all components of the Jacobians. We collect them below.

$$J_{s1}(\theta) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{s2}(\theta) = \begin{bmatrix} c_1 \\ s_1 \\ 0 \\ -L_1 s_1 \\ L_1 c_1 \\ 0 \end{bmatrix}$$

$$J_{s3}(\theta) = \begin{bmatrix} s_1 s_2 \\ -c_1 s_2 \\ c_2 \\ L_1 c_1 s_2 \\ L_1 s_1 s_2 \\ 0 \end{bmatrix}$$

$$J_{s4}(\theta) = \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 \\ s_1 c_3 + c_1 c_2 s_3 \\ s_2 s_3 \\ -L_1(s_1 c_3 + c_1 c_2 s_3) - L_2(s_1 c_2 c_3 + c_1 s_3) \\ L_1(c_1 c_3 - s_1 c_2 s_3) + L_2(c_1 c_2 c_3 - s_1 s_3) \\ L_2 s_2 c_3 \end{bmatrix}$$

$$J_{s5}(\theta) = \begin{bmatrix} \omega_5 \\ v_5 \end{bmatrix}$$

$$\omega_5 = \begin{bmatrix} c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix}$$

$$v_5 = -\begin{bmatrix} c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix} \times \begin{bmatrix} L_2 s_1 s_2 \\ -L_2 c_1 s_2 \\ L_1 + L_2 c_2 \end{bmatrix}$$

$$J_{s6}(\theta) = \begin{bmatrix} \omega_6 \\ v_6 \end{bmatrix}$$

$$\omega_6 = \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 & -c_1 s_3 c_4 - s_1 c_2 c_3 c_4 + s_1 s_2 s_4 & c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 c_3 + c_1 c_2 s_3 & -s_1 s_3 c_4 + c_1 c_2 c_3 c_4 - c_1 s_2 s_4 & s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ s_2 s_3 & s_2 c_3 c_4 + c_2 s_4 & -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix} \begin{bmatrix} c_5 \\ s_5 \\ 0 \end{bmatrix}$$

$$v_6 = -\omega_6 \times \begin{bmatrix} L_2 s_1 s_2 + L_3 s_3 s_4 \\ -L_2 c_1 s_2 - L_3 c_3 s_4 \\ L_1 + L_2 c_2 + L_3 c_4 \end{bmatrix}$$

$$J_{s7}(\theta) = \begin{bmatrix} \omega_7 \\ v_7 \end{bmatrix}$$

$$\omega_7 = \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3c_4 - s_1c_2c_3c_4 + s_1s_2s_4 & c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1c_3 + c_1c_2s_3 & -s_1s_3c_4 + c_1c_2c_3c_4 - c_1s_2s_4 & s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ s_2s_3 & s_2c_3c_4 + c_2s_4 & -s_2c_3s_4 + c_2c_4 \end{bmatrix} \begin{bmatrix} s_5s_6 \\ -c_5s_6 \\ c_6 \end{bmatrix}$$

$$v_7 = -\omega_7 \times \begin{bmatrix} L_2s_1s_2 + L_3s_3s_4 \\ -L_2c_1s_2 - L_3c_3s_4 \\ L_1 + L_2c_2 + L_3c_4 \end{bmatrix}$$

At zero configuration, $\theta = (\theta_i = 0)$, $s_i = 0$ and $c_i = 1$. We obtain:

$$J_{s1}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{s2}(0) = \begin{bmatrix} c_1 \\ s_1 \\ 0 \\ -L_1s_1 \\ L_1c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ L_1 \\ 0 \end{bmatrix}$$

$$J_{s3}(0) = \begin{bmatrix} s_1s_2 \\ -c_1s_2 \\ c_2 \\ L_1c_1s_2 \\ L_1s_1s_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{s4}(0) = \begin{bmatrix} c_1c_3 - s_1c_2s_3 \\ s_1c_3 + c_1c_2s_3 \\ s_2s_3 \\ -L_1(s_1c_3 + c_1c_2s_3) - L_2(s_1c_2c_3 + c_1s_3) \\ L_1(c_1c_3 - s_1c_2s_3) + L_2(c_1c_2c_3 - s_1s_3) \\ L_2s_2c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ L_1 + L_2 \\ 0 \end{bmatrix}$$

$$\omega_5 = \begin{bmatrix} c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ -s_2c_3s_4 + c_2c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_5 = -\omega_5 \times \begin{bmatrix} L_2s_1s_2 \\ -L_2c_1s_2 \\ L_1 + L_2c_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 + L_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{s5}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_6 = \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3c_4 - s_1c_2c_3c_4 + s_1s_2s_4 & c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1c_3 + c_1c_2s_3 & -s_1s_3c_4 + c_1c_2c_3c_4 - c_1s_2s_4 & s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ s_2s_3 & s_2c_3c_4 + c_2s_4 & -s_2c_3s_4 + c_2c_4 \end{bmatrix} \begin{bmatrix} c_5 \\ s_5 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_6 = -\omega_6 \times \begin{bmatrix} L_2s_1s_2 + L_3s_3s_4 \\ -L_2c_1s_2 - L_3c_3s_4 \\ L_1 + L_2c_2 + L_3c_4 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 + L_2 + L_3 \end{bmatrix} = \begin{bmatrix} 0 \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix}$$

$$J_{s6}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ L_1 + L_2 + L_3 \\ 0 \end{bmatrix}$$

$$\omega_7 = \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3c_4 - s_1c_2c_3c_4 + s_1s_2s_4 & c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1c_3 + c_1c_2s_3 & -s_1s_3c_4 + c_1c_2c_3c_4 - c_1s_2s_4 & s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ s_2s_3 & s_2c_3c_4 + c_2s_4 & -s_2c_3s_4 + c_2c_4 \end{bmatrix} \begin{bmatrix} s_5s_6 \\ -c_5s_6 \\ c_6 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_7 = -\omega_7 \times \begin{bmatrix} L_2s_1s_2 + L_3s_3s_4 \\ -L_2c_1s_2 - L_3c_3s_4 \\ L_1 + L_2c_2 + L_3c_4 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ L_1 + L_2 + L_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{s7}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We obtained the same result as we found for question 1(a) without deriving the general expressions for the Jacobians. Hopefully this kind of assures us that the general expressions for the Jacobians are correct.

Now move on to find the answer for question 1(b).

Let $\boldsymbol{\theta_t}$ be the given angles, where each joint $\theta_i = {}^{i\pi}/_{16}$ radians, we have

$$\begin{aligned} \boldsymbol{\theta_t} &= (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7) \\ &= \left(\frac{\pi}{16}, \frac{2\pi}{16}, \frac{3\pi}{16}, \frac{4\pi}{16}, \frac{5\pi}{16}, \frac{6\pi}{16}, \frac{7\pi}{16}\right) \\ &= \left(\frac{\pi}{16}, \frac{\pi}{8}, \frac{3\pi}{16}, \frac{\pi}{4}, \frac{5\pi}{16}, \frac{3\pi}{8}, \frac{7\pi}{16}\right) \end{aligned}$$

Then

i	1	2	3	4	5	6	7
$s_i = \sin\theta_i$	0.1951	0.3827	0.5556	0.7071	0.8315	0.9239	0.9808
$c_i = \cos\theta_i$	0.9808	0.9239	0.8315	0.7071	0.5556	0.3827	0.1951

$$J_{s1}(\boldsymbol{\theta_t}) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$J_{s2}(\boldsymbol{\theta_t}) = \begin{bmatrix} c_1 \\ s_1 \\ 0 \\ -L_1 s_1 \\ L_1 c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.9808 \\ 0.1951 \\ 0 \\ -0.0663 \\ 0.3335 \\ 0 \end{bmatrix}$$

$$J_{s3}(\boldsymbol{\theta_t}) = \begin{bmatrix} s_1 s_2 \\ -c_1 s_2 \\ c_2 \\ L_1 c_1 s_2 \\ L_1 s_1 s_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0747 \\ -0.3753 \\ 0.9239 \\ 0.1276 \\ 0.0254 \\ 0 \end{bmatrix}$$

$$J_{s4}(\boldsymbol{\theta_t}) = \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 \\ s_1 c_3 + c_1 c_2 s_3 \\ s_2 s_3 \\ -L_1(s_1 c_3 + c_1 c_2 s_3) - L_2(s_1 c_2 c_3 + c_1 s_3) \\ L_1(c_1 c_3 - s_1 c_2 s_3) + L_2(c_1 c_2 c_3 - s_1 s_3) \\ L_2 s_2 c_3 \end{bmatrix} = \begin{bmatrix} 0.7154 \\ 0.6656 \\ 0.2126 \\ -0.5042 \\ 0.5012 \\ 0.1227 \end{bmatrix}$$

$$\omega_5 = \begin{bmatrix} c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix} = \begin{bmatrix} 0.5441 \\ -0.7215 \\ 0.4283 \end{bmatrix}$$

$$v_5 = -\omega_5 \times \begin{bmatrix} L_2 s_1 s_2 \\ -L_2 c_1 s_2 \\ L_1 + L_2 c_2 \end{bmatrix} = -\begin{bmatrix} 0.5441 \\ -0.7215 \\ 0.4283 \end{bmatrix} \times \begin{bmatrix} 0.0299 \\ -0.1501 \\ 0.7096 \end{bmatrix} = \begin{bmatrix} -0.4477 \\ -0.3732 \\ -0.0601 \end{bmatrix}$$

$$J_{s5}(\boldsymbol{\theta_t}) = \begin{bmatrix} 0.5441 \\ -0.7215 \\ 0.4283 \\ -0.4477 \\ -0.3732 \\ -0.0601 \end{bmatrix}$$

$$\omega_6 = \begin{bmatrix} c_1 c_3 - s_1 c_2 s_3 & -c_1 s_3 c_4 - s_1 c_2 c_3 c_4 + s_1 s_2 s_4 & c_1 s_3 s_4 + s_1 c_2 c_3 s_4 + s_1 s_2 c_4 \\ s_1 c_3 + c_1 c_2 s_3 & -s_1 s_3 c_4 + c_1 c_2 c_3 c_4 - c_1 s_2 s_4 & s_1 s_3 s_4 - c_1 c_2 c_3 s_4 - c_1 s_2 c_4 \\ s_2 s_3 & s_2 c_3 c_4 + c_2 s_4 & -s_2 c_3 s_4 + c_2 c_4 \end{bmatrix} \begin{bmatrix} c_5 \\ s_5 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0.7154 & -0.4385 & 0.5441 \\ 0.6656 & 0.1907 & -0.7215 \\ 0.1084 & 0.8783 & 0.4283 \end{bmatrix} \begin{bmatrix} 0.5556 \\ 0.8315 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0329 \\ 0.5284 \\ 0.7905 \end{bmatrix}$$

$$v_6 = -\omega_6 \times \begin{bmatrix} L_2 s_1 s_2 + L_3 s_3 s_4 \\ -L_2 c_1 s_2 - L_3 c_3 s_4 \\ L_1 + L_2 c_2 + L_3 c_4 \end{bmatrix} = -\begin{bmatrix} 0.0329 \\ 0.5284 \\ 0.7905 \end{bmatrix} \times \begin{bmatrix} 0.1870 \\ -0.3766 \\ 0.9924 \end{bmatrix} = \begin{bmatrix} 0.8221 \\ 0.1152 \\ -0.1112 \end{bmatrix}$$

$$J_{s6}(\boldsymbol{\theta_t}) = \begin{bmatrix} 0.0329 \\ 0.5284 \\ 0.7905 \\ 0.8221 \\ 0.1152 \\ -0.1112 \end{bmatrix}$$

$$\omega_7 = \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_1s_3c_4 - s_1c_2c_3c_4 + s_1s_2s_4 & c_1s_3s_4 + s_1c_2c_3s_4 + s_1s_2c_4 \\ s_1c_3 + c_1c_2s_3 & -s_1s_3c_4 + c_1c_2c_3c_4 - c_1s_2s_4 & s_1s_3s_4 - c_1c_2c_3s_4 - c_1s_2c_4 \\ s_2s_3 & s_2c_3c_4 + c_2s_4 & -s_2c_3s_4 + c_2c_4 \end{bmatrix} \begin{bmatrix} s_5s_6 \\ -c_5s_6 \\ c_6 \end{bmatrix}$$

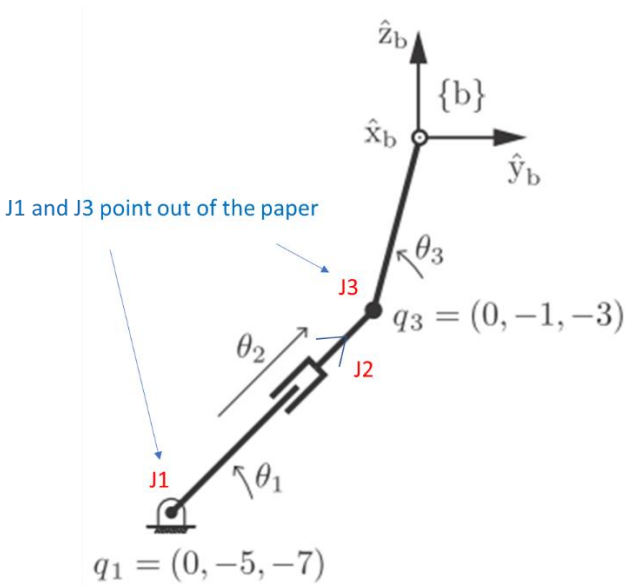
$$= \begin{bmatrix} 0.7154 & -0.4385 & 0.5441 \\ 0.6656 & 0.1907 & -0.7215 \\ 0.1084 & 0.8783 & 0.4283 \end{bmatrix} \begin{bmatrix} 0.7682 \\ -0.5133 \\ 0.3827 \end{bmatrix} = \begin{bmatrix} 0.9828 \\ 0.1373 \\ -0.2036 \end{bmatrix}$$

$$v_7 = -\omega_7 \times \begin{bmatrix} L_2s_1s_2 + L_3s_3s_4 \\ -L_2c_1s_2 - L_3c_3s_4 \\ L_1 + L_2c_2 + L_3c_4 \end{bmatrix} = -\begin{bmatrix} 0.9828 \\ 0.1373 \\ -0.2036 \end{bmatrix} \times \begin{bmatrix} 0.1870 \\ -0.3766 \\ 0.9924 \end{bmatrix} = \begin{bmatrix} 0.0596 \\ -1.0134 \\ -0.3958 \end{bmatrix}$$

$$J_{s7}(\boldsymbol{\theta_t}) = \begin{bmatrix} 0.9828 \\ 0.1373 \\ -0.2036 \\ 0.0596 \\ -1.0134 \\ -0.3958 \end{bmatrix}$$

Question 2: RPR robot arm

(Exercise 5.3)



There are three joints, n = 3, the Jacobian matrix has 3 columns.

$$J_b(\theta) = [J_{b1} \quad J_{b2} \quad J_{b3}]$$

Since we are looking for body Jacobian, we start from the joint closest to the {b} frame and progress to joints further from {b}, i.e. Joint 3, then 2, then 1. We take note the Jacobian column of a joint will

be affected by the motion of "preceding" joints. In the case of body Jacobian, "preceding" means closer to the {b} frame.

From the diagram, we can see that ω_1 (J1) and ω_3 (J3) are in the direction of $\hat{x}_b$ (note the reference is {b} frame) taking note of the direction of θ_1 and θ_3 in RHR. We also note J2 is a prismatic joint at a direction along the first link.

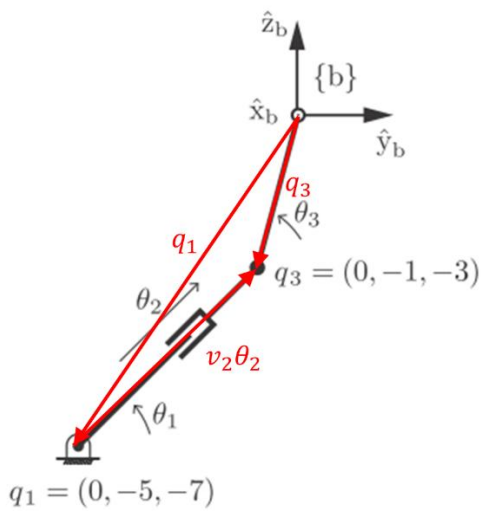
We now find $J_{b3} = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix}$. We already note $\omega_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ since J3 is in the direction of $\hat{x}_b$. We have been given the vector q3 connecting J3 and {b}, we find

$$v_3 = -\omega_3 \times q_3 = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Next we find $J_{b2} = \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix}$.

J2 is a prismatic joint, so $\omega_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. v_2 is the direction (with reference to {b}) of the prismatic joint. We can see that it changes (with reference to {b}) when θ_3 change. It should be a function of θ_3. However for the question, θ_3 is not given, and the question ask for the answer in numerical, i.e. not in a function of θ_3, for e.g. Instead we know q_1 and q_3. We can find v_2 from vector addition of q_1 and q_3.

From the diagram below, we can see that vector of the linear translation by θ_2 is given by the vector $v_2\theta_2$ where v_2 is the unit vector that gives the direction of the translation and θ_2 is gives the magnitude of the translation.



$$q_1 = q_3 + (-v_2\theta_2)$$

$$v_2\theta_2 = q_3 - q_1 = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ -5 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$$

To find v_2, we divide $v_2\theta_2$ by its magnitude $\|v_2\theta_2\|$ to get the direction (unit vector).

$$\|v_2\theta_2\| = \sqrt{0 + 4^2 + 4^2} = \sqrt{2 \times 4^2} = \sqrt{2}\sqrt{4^2} = 4\sqrt{2}$$

$$v_2 = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \div 4\sqrt{2} = \begin{bmatrix} 0 \\ 4/4\sqrt{2} \\ 4/4\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Lastly, we find $J_{b1} = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix}$. This is similar to J_{b3}.

$$\omega_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = -\omega_1 \times q_1 = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -5 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ 5 \end{bmatrix}$$

Putting all together,

$$J_b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -7 & 1/\sqrt{2} & -3 \\ 5 & 1/\sqrt{2} & 1 \end{bmatrix}$$